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A NOTE ON DEGENERATE FUBINI POLYNOMIALS

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ABSTRACT. As degenerate versions of Fubini and two variable Fubini polynomials, we introduce the degenerate Fubini and the two variable degenerate Fubini polynomials. We investigate their properties, recurrence relations and explicit formulas for those polynomials.

1. Introduction

It is well knwon that the Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \ (n \geq 0), \quad (\text{see } [4,6,10,11,12,13]). \tag{1.1}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$. Alternatively, the Stirling numbers of the second kind are given by the generating function

$$\frac{1}{k!}(e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \ (k \geq 0), \quad (\text{see } [7,9]). \tag{1.2}$$

The Stirling numbers of the first kind are defined by the generating function

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \ (k \geq 0), \quad (\text{see } [5,8,10]). \tag{1.3}$$

In [1,2], L. Carlitz considered the degenerate Bernoulli numbers defined by

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!}, \ (\lambda \in \mathbb{R}). \tag{1.4}$$

From (1.4), we note that $\lim_{\lambda \to 0} \beta_{n,\lambda}(x) = B_n(x)$, where $B_n(x)$ are the ordinary Bernoulli polynomials given by $\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$.

Now, we observe that

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$$(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \left(\frac{x}{\lambda}\right)_n \lambda^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad \text{(see [6, 9, 10])}. \tag{1.5}$$

where $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$, $(n \geq 1)$, $(x)_{0,\lambda} = 1$.

In view of (1.2) and (1.5), we consider the degenerate Stirling numbers of the second kind given by the generating function

$$\frac{1}{k!}((1+\lambda t)^{\frac{1}{\lambda}} - 1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \quad \text{(see [9])}. \tag{1.6}$$

From (1.2) and (1.6), we note that $\lim_{\lambda\to 0} S_{2,\lambda}(n,k) = S_2(n,k)$. By (1.6), we easily get

$$\begin{aligned}\frac{1}{k!}((1+\lambda t)^{\frac{1}{\lambda}} - 1)^k &= \frac{1}{k!}(e^{\frac{1}{\lambda}\log(1+\lambda t)} - 1)^k = \sum_{n=k}^{\infty} S_2(n,k)\lambda^{-n}\frac{(\log(1+\lambda t))^n}{n!} \\ &= \sum_{n=k}^{\infty} S_2(n,k)\lambda^{-n} \sum_{m=n}^{\infty} S_1(m,n)\frac{\lambda^m t^m}{m!} \\ &= \sum_{m=k}^{\infty}\left(\sum_{n=k}^{m} S_2(n,k)\lambda^{m-n} S_1(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{1.7}$$

From (1.6) and (1.7), we have

$$S_{2,\lambda}(m,k) = \sum_{n=k}^{m} \lambda^{m-n} S_2(n,k) S_1(m,n), \ (m,k \geq 0). \tag{1.8}$$

As is well known, Fubini polynomials are defined as

$$F_n(y) = \sum_{k=0}^{n} S_2(n,k)k!y^k, \quad \text{(see [5, 7])}. \tag{1.9}$$

The generating function of Fubini polynomials is given by

$$\frac{1}{1-y(e^t-1)} = \sum_{n=0}^{\infty} F_n(y)\frac{t^n}{n!}, \quad \text{(see [5])}, \tag{1.10}$$

which is related to the geometric series

$$\left(y\frac{d}{dy}\right)^m\left(\frac{1}{1-y}\right)=\sum_{k=0}^{\infty}k^m y^k=\frac{1}{1-y}F_m(\frac{y}{1-y}). \tag{1.11}$$

The nth Fubini number(or ordered Bell number) is defined as

$$F_n(1)=F_n=\sum_{k=0}^{n}S_2(n,k)k!, \quad \text{(see [3, 5, 6, 7])}, \tag{1.12}$$

and counts all the possible set partitions of a set with n elements such that the order of the blocks matters.

From (1.11), we see that

$$F_n=\frac{1}{2}\sum_{k=0}^{\infty}\frac{k^n}{2^k}, \quad \text{(see [3, 5, 7])}. \tag{1.13}$$

In [5], L. Kargin considered two variable Fubini polynomials and gave some explicit formulas for those polynomials.

In this paper, we consider the degenerate Fubini and the two variable degenerate Fubini polynomials with the viewpoint of Carlitz. We investigate some properties, recurrence relations and explicit formulas for those polynomials.

2. Degenerate Fubini polynomials

In view of (1.10), we define the degenerate Fubini polynomials as

$$\frac{1}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}=\sum_{n=0}^{\infty}F_{n,\lambda}(y)\frac{t^n}{n!}, \ (\lambda\in\mathbb{R}). \tag{2.1}$$

Note that $\lim_{\lambda\to 0}F_{n,\lambda}(y)=F_n(y)$.

From the left side of (2.1), we note that

$$\begin{aligned}\frac{1}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}&=\sum_{k=0}^{\infty}y^k((1+\lambda t)^{\frac{1}{\lambda}}-1)^k\\&=\sum_{k=0}^{\infty}y^k k!\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}y^k k!S_{2,\lambda}(n,k)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.2}$$

Therefore, by (2.1) and (2.2), we obtain the following theorem.

Theorem 2.1. *For $n\geq 0$, we have*

$$F_{n,\lambda}(y)=\sum_{k=0}^{n}y^k k!S_{2,\lambda}(n,k).$$

The degenerate Stirling number of the second kind is given by (1.7) to be

$$\begin{aligned}\frac{1}{k!}((1+\lambda t)^{\frac{1}{\lambda}}-1)^k &= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(1+\lambda t)^{\frac{l}{\lambda}} \\ &= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\sum_{n=0}^{\infty}\left(\frac{l}{\lambda}\right)_n \lambda^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(l)_{n,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

Thus, by (1.6) and (2.3), we get

$$\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(l)_{n,\lambda} = \begin{cases} 0, & \text{if } n<k \\ S_{2,\lambda}(n,k), & \text{if } n \geq k. \end{cases} \tag{2.4}$$

Now, we observe that

$$\begin{aligned}\frac{1}{1-y}\left(\frac{1}{1-\frac{y}{1-y}((1+\lambda t)^{\frac{1}{\lambda}}-1)}\right) &= \frac{1}{1-y(1+\lambda t)^{\frac{1}{\lambda}}} = \sum_{k=0}^{\infty} y^k (1+\lambda t)^{\frac{k}{\lambda}} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} y^k (k)_{n,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.5}$$

On the other hand, by (1.7), we get

$$\frac{1}{1-y}\left(\frac{1}{1-\frac{y}{1-y}((1+\lambda t)^{\frac{1}{\lambda}}-1)}\right) = \frac{1}{1-y}\sum_{n=0}^{\infty} F_{n,\lambda}\left(\frac{y}{1-y}\right)\frac{t^n}{n!}. \tag{2.6}$$

Therefore, by (2.5) and (2.6), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$\frac{1}{1-y}F_{n,\lambda}\left(\frac{y}{1-y}\right) = \sum_{k=0}^{\infty} y^k (k)_{n,\lambda}.$$

Note that

$$F_{n,\lambda}(1) = F_{n,\lambda} = \sum_{k=0}^{n} k! S_{2,\lambda}(n,k), \tag{2.7}$$

and, from Theorem 2.2, we have

$$F_{n,\lambda} = \frac{1}{2}\sum_{k=0}^{\infty}\frac{(k)_{n,\lambda}}{2^k}. \tag{2.8}$$

3. Two variable degenerate Fubini polynomials

Here we define the two variable degenerate Fubini polynomials as

$$\sum_{n=0}^{\infty} F_{n,\lambda}(x;y)\frac{t^n}{n!} = \frac{1}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}}. \tag{3.1}$$

When $x=0$, $F_{n,\lambda}(0;y)=F_{n,\lambda}(y)$, $F_{n,\lambda}(0;1)=F_{n,\lambda}$. From (3.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n,\lambda}(x;y)\frac{t^n}{n!} &= \frac{1}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \left(\sum_{l=0}^{\infty} F_{l,\lambda}(y)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{k,\lambda}(y)(x)_{n-k,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{3.2}$$

Comparing the coefficients on both sides of (3.2), we obtain the following theorem.

Theorem 3.1. *For $n \geq 0$, we have*

$$F_{n,\lambda}(x;y) = \sum_{k=0}^{n}\binom{n}{k}F_{k,\lambda}(y)(x)_{n-k,\lambda}.$$

Now, we observe that

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(F_{n,\lambda}(x+1;y)-F_{n,\lambda}(x;y)\right)\frac{t^n}{n!} = \frac{(1+\lambda t)^{\frac{x}{\lambda}}((1+\lambda t)^{\frac{1}{\lambda}}-1)}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)} \\ &= \frac{1}{y}\frac{y((1+\lambda t)^{\frac{1}{\lambda}}-1)}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}} = \frac{1}{y}\left\{\frac{(1+\lambda t)^{\frac{x}{\lambda}}}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}-(1+\lambda t)^{\frac{x}{\lambda}}\right\} \\ &= \frac{1}{y}\sum_{n=0}^{\infty}\left(F_{n,\lambda}(x;y)-(x)_{n,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{3.3}$$

Comparing the coefficients on both sides of (3.3), we have

$$\begin{aligned}yF_{n,\lambda}(x+1;y) &= yF_{n,\lambda}(x;y)+F_{n,\lambda}(x;y)-(x)_{n,\lambda} \\ &= (y+1)F_{n,\lambda}(x;y)-(x)_{n,\lambda},\ (n\geq 0).\end{aligned} \tag{3.4}$$

When $x=0$ and $x=-1$ in (3.4), respectively we have

$$yF_{n,\lambda}(1;y) = (y+1)F_{n,\lambda}(y),\ (n>0), \tag{3.5}$$

and

$$yF_{n,\lambda}(y) = (y+1)F_{n,\lambda}(-1;y) - (-1)_{n,\lambda}, \ (n \geq 0). \tag{3.6}$$

Therefore, by (3.4), (3.5) and (3.6), we obtain the following theorem.

Theorem 3.2. *For $n \geq 0$, we have*

$$yF_{n,\lambda}(x+1;y) = (y+1)F_{n,\lambda}(x;y) - (x)_{n,\lambda}.$$

In particular,

$$yF_{n,\lambda}(1;y) = (y+1)F_{n,\lambda}(y), \ (n > 0),$$

and

$$yF_{n,\lambda}(y) = (y+1)F_{n,\lambda}(-1;y) - (-1)_{n,\lambda}, \ (n \geq 0).$$

It is not difficult to show that

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{(1+\lambda t)^{\frac{x}{\lambda}}}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\right) \\
&= \frac{x(1+\lambda t)^{\frac{x-\lambda}{\lambda}}}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)} + \left(\frac{y(1+\lambda t)^{\frac{1-\lambda}{\lambda}}}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\right)\left(\frac{(1+\lambda t)^{\frac{x}{\lambda}}}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\right).
\end{aligned} \tag{3.7}$$

Let us take $x = x_1 + x_2 + \lambda - 1$ in (3.7). Then we have

$$\frac{\partial}{\partial t}\left(\frac{1}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x_1+x_2+\lambda-1}{\lambda}}\right) = \sum_{n=0}^{\infty} F_{n+1,\lambda}(x_1+x_2+\lambda-1;y)\frac{t^n}{n!}, \tag{3.8}$$

$$\frac{(x_1+x_2+\lambda-1)(1+\lambda t)^{\frac{x_1+x_2-1}{\lambda}}}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)} = (x_1+x_2+\lambda-1)\sum_{n=0}^{\infty} F_{n,\lambda}(x_1+x_2-1;y)\frac{t^n}{n!}, \tag{3.9}$$

and

$$\begin{aligned}
&\left(\frac{y(1+\lambda t)^{\frac{1-\lambda}{\lambda}}}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\right)\left(\frac{(1+\lambda t)^{\frac{x_1+x_2+\lambda-1}{\lambda}}}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\right) \\
&= y\left(\sum_{l=0}^{\infty} F_{l,\lambda}(x_1;\lambda)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} F_{m,\lambda}(x_2;y)\frac{t^m}{m!}\right) \\
&= y\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{k,\lambda}(x_1;y)F_{n-k,\lambda}(x_2;y)\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.10}$$

From (3.7), (3.8), (3.9) and (3.10), we note that

$$\begin{aligned} & y\sum_{k=0}^{n}\binom{n}{k}F_{k,\lambda}(x_1;y)F_{n-k,\lambda}(x_2;y) \\ &= F_{n+1,\lambda}(x_1+x_2+\lambda-1;y)-(x_1+x_2+\lambda-1)F_{n,\lambda}(x_1+x_2-1;y). \end{aligned} \tag{3.11}$$

Let us take $x_1 = x_2 = 0$ in (3.11). Then we have

$$y\sum_{k=0}^{n}\binom{n}{k}F_{k,\lambda}(y)F_{n-k,\lambda}(y) = F_{n+1,\lambda}(\lambda-1;y)-(\lambda-1)F_{n,\lambda}(-1;y). \tag{3.12}$$

From Theorem 3.2, we can easily derive the following equations (3.13)

$$yF_{n,\lambda}(x+\lambda+1;y) = (y+1)F_{n,\lambda}(x+\lambda;y)-(x+\lambda)_{n,\lambda}, \tag{3.13}$$

and

$$yF_{n,\lambda}(\lambda;y) = (y+1)F_{n,\lambda}(\lambda-1;y)-(\lambda-1)_{n,\lambda},\ (n\geq 0). \tag{3.14}$$

Thus, by (3.14), we get

$$(y+1)F_{n,\lambda}(\lambda-1;y) = yF_{n,\lambda}(\lambda;y)+(\lambda-1)_{n,\lambda},\ (n\geq 0). \tag{3.15}$$

From (3.12) and (3.15), we have

$$\begin{aligned} & y\sum_{k=0}^{n}\binom{n}{k}F_{k,\lambda}(y)F_{n-k,\lambda}(y) \\ &= \frac{y}{y+1}F_{n+1,\lambda}(\lambda;y)+\frac{1}{y+1}(\lambda-1)_{n+1,\lambda} \\ &\quad -\left\{\frac{y(\lambda-1)}{y+1}F_{n,\lambda}(\lambda;y)+\frac{1}{y+1}(\lambda-1)(-1)_{n,\lambda}\right\} \\ &= \frac{y}{y+1}\left(F_{n+1,\lambda}(\lambda;y)-(\lambda-1)F_{n,\lambda}(\lambda;y)\right). \end{aligned} \tag{3.16}$$

Therefore, by (3.16), we obtain the following theorem.

Theorem 3.3. *For $n \geq 0$, we have*

$$\begin{aligned} & (y+1)\sum_{k=0}^{n}\binom{n}{k}F_{k,\lambda}(y)F_{n-k,\lambda}(y) \\ &= F_{n+1,\lambda}(\lambda;y)-(\lambda-1)F_{n,\lambda}(\lambda;y). \end{aligned}$$

When $y = 1$ in Theorem 5, we have

$$2\sum_{k=0}^{n}\binom{n}{k}F_{k,\lambda}F_{n-k,\lambda} = F_{n+1,\lambda}(\lambda;1)-(\lambda-1)F_{n,\lambda}(\lambda;1). \tag{3.17}$$

Now, we observe that

$$\left(\frac{(1+\lambda t)^{\frac{x_1}{\lambda}}}{1-y_1((1+\lambda t)^{\frac{1}{\lambda}}-1)}\right)\left(\frac{(1+\lambda t)^{\frac{x_2}{\lambda}}}{1-y_2((1+\lambda t)^{\frac{1}{\lambda}}-1)}\right)$$
$$=\frac{y_2}{y_2-y_1}\left(\frac{(1+\lambda t)^{\frac{x_1+x_2}{\lambda}}}{1-y_2((1+\lambda t)^{\frac{1}{\lambda}}-1)}\right)-\left(\frac{y_1}{y_2-y_1}\right)\left(\frac{(1+\lambda t)^{\frac{x_1+x_2}{\lambda}}}{1-y_1((1+\lambda t)^{\frac{1}{\lambda}}-1)}\right) \tag{3.18}$$

From (3.1) and (3.18), we have

$$\begin{aligned}&\left(\sum_{l=0}^{\infty}F_{l,\lambda}(x_1;y_1)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}F_{m,\lambda}(x_2;y_2)\frac{t^m}{m!}\right)\\&=\frac{1}{y_2-y_1}\left\{\sum_{n=0}^{\infty}y_2F_{n,\lambda}(x_1+x_2;y_2)\frac{t^n}{n!}-\sum_{n=0}^{\infty}y_1F_{n,\lambda}(x_1+x_2;y_1)\frac{t^n}{n!}\right\}\\&=\frac{1}{y_2-y_1}\sum_{n=0}^{\infty}\left(y_2F_{n,\lambda}(x_1+x_2;y_2)-y_1F_{n,\lambda}(x_1+x_2;y_1)\right)\frac{t^n}{n!}.\end{aligned} \tag{3.19}$$

On the other hand,

$$\begin{aligned}&\left(\sum_{l=0}^{\infty}F_{l,\lambda}(x_1;y_1)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}F_{m,\lambda}(x_2;y_2)\frac{t^m}{m!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{k,\lambda}(x_1;y_1)F_{n-k,\lambda}(x_2;y_2)\right)\frac{t^n}{n!}.\end{aligned} \tag{3.20}$$

Therefore, by (3.19) and (3.20), we obtain the following theorem.

Theorem 3.4. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}F_{k,\lambda}(x_1;y_1)F_{n-k,\lambda}(x_2;y_2)=\frac{y_2F_{n,\lambda}(x_1+x_2;y_2)-y_1F_{n,\lambda}(x_1+x_2;y_1)}{y_2-y_1}.$$

In particular, for $x_1=x_2=0$ in Theorem 3.4, we have

$$\sum_{k=0}^{n}\binom{n}{k}F_{k,\lambda}(y_1)F_{n-k,\lambda}(y_2)=\frac{y_2F_{n,\lambda}(y_2)-y_1F_{n,\lambda}(y_1)}{y_2-y_1}. \tag{3.21}$$

Therefore, by (3.21), we obtain the following corollary.

Corollary 3.5. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} F_{k,\lambda}(y_1) F_{n-k,\lambda}(y_2) = \frac{y_2 F_{n,\lambda}(y_2) - y_1 F_{n,\lambda}(y_1)}{y_2 - y_1}.$$

From (2.1), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} F_{n,\lambda}(x; y-1) \frac{t^n}{n!} &= \frac{(1+\lambda t)^{\frac{x}{\lambda}}}{1-(y-1)((1+\lambda t)^{\frac{1}{\lambda}} - 1)} \\ &= \frac{(1+\lambda t)^{\frac{x}{\lambda}}}{1 - y((1+\lambda t)^{\frac{1}{\lambda}} - 1) + (1+\lambda t)^{\frac{1}{\lambda}} - 1} \\ &= \frac{(1+\lambda t)^{\frac{x-1}{\lambda}}}{1 - y((1-\lambda t)^{\frac{1}{\lambda}})} \\ &= \frac{(1-\lambda(-t))^{-\frac{1-x}{\lambda}}}{1 + y((1-\lambda(-t))^{-\frac{1}{\lambda}} - 1)} = \sum_{n=0}^{\infty} F_{n,-\lambda}(1-x; -y) \frac{(-1)^n}{n!} t^n. \end{aligned} \tag{3.22}$$

Comparing the coefficients on both sides of (3.22), we obtain the following theorem.

Theorem 3.6. *For $n \geq 0$, we have*

$$F_{n,\lambda}(x; y-1) = (-1)^n F_{n,-\lambda}(1-x; -y).$$

When $x = 0$ in Theorem 3.6, we have

$$F_{n,\lambda}(y-1) = (-1)^n F_{n,-\lambda}(1; -y) = (-1)^n \left(\frac{y-1}{y} \right) F_{n,-\lambda}(-y), \ (n > 0). \tag{3.23}$$

Replacing y by $y+1$ in (3.23), we get

$$F_{n,\lambda}(y) = (-1)^n \frac{y}{y+1} F_{n,-\lambda}(-1-y), \ (n > 0). \tag{3.24}$$

From (3.24), we note that

$$\begin{aligned} F_{n,\lambda}(y) &= (-1)^n \frac{y}{y+1} \sum_{k=0}^{n} k!(-1)^k (y+1)^k S_{2,-\lambda}(n,k) \\ &= y \sum_{k=0}^{n} k!(-1)^{n+k} (y+1)^{k-1} S_{2,-\lambda}(n,k), \ (n > 0). \end{aligned} \tag{3.25}$$

Therefore, by (3.25), we obtain the following theorem.

Theorem 3.7. *For $n \in \mathbb{N}$, we have*

$$F_{n,\lambda}(y) = y \sum_{k=0}^{n} k!(-1)^{n+k}(y+1)^{k-1} S_{2,-\lambda}(n,k).$$

Let us take $y=-2$ in (3.24). Then we get

$$F_{n,\lambda}(-2) = 2(-1)^n F_{n,-\lambda}(1) = 2(-1)^n F_{n,-\lambda}. \tag{3.26}$$

Let us take $y_1=-2$ and $y_2=1$ in Corollary 3.5, and then invoke Theorem 3.6. Then we have

$$2\sum_{k=0}^{n} \binom{n}{k} F_{k,-\lambda}(-1)^k F_{n-k,\lambda} = \frac{1}{3}(F_{n,\lambda} + (-1)^n 4F_{n,-\lambda}). \tag{3.27}$$

Therefore, by (3.27), we obtain the following theorem.

Theorem 3.8. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{k,-\lambda} F_{n-k,-\lambda} = \frac{1}{6}(F_{n,\lambda} + (-1)^n 4F_{n,-\lambda})$$

References

1. L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers,* Utilitas Math. **15** (1979), 51-88.
2. L. Carlitz, *A degenerate Staudt-Clausen theorem,* Arch. Math. (Basel) **7** (1956), 28-33.
3. T. Diagana, H. Maíga, *Some new identities and congruences for Fubini numbers,* J. Number Theory **173** (2017), 547-569.
4. T. Ernst, *q-Leibniz functional matrices with applications to q-Pascal and q-Stirling matrices,* Adv. Stud. Contemp. Math. (Kyungshang) **22** (2012), no. 4, 537-555.
5. L. Kargin, *Some formulae for products of Fubini polynomials with applications,* arXiv:1701.01023 v1[math. CA], 23 Dec. 2016.
6. G.-W. Jang, T. Kim, *Some identities of ordered Bell numbers arising from differential equations,* Adv. Stud. Contemp. Math. **27** (2017), no. 3, 385-397.
7. T. Kim, *Degenerate ordered Bell numbers and polynomials,* Proc. Jangjeon Math. Soc. **20** (2017), no. 2, 137-144.
8. T. Kim, D. S. Kim, G.-W. Jang, *Extended stirling polynomials of the second kind and extended Bell polynomials,* Proc. Jangjeon Math. Soc. **20** (2017), no. 3, 365-376.
9. T. Kim, *A note on degenerate stirling polynomials of the second kind,* Proc. Jangjeon Math. Soc. **20** (2017), no. 3, 319-331.
10. T. Kim, *λ-analogue of stirling numbers of the first kind,* Adv. Stud. Contemp. Math. **27** (2017), no. 3, 423-429.
11. A. K. Kwaśniewski, *Lucky 13-th exercises on Stirling-like numbers and Dobinski-like formula,* Proc. Jangjeon Math. Soc. **11** (2008), no. 2, 137-144.
12. A. K. Kwaśniewski, *On ψ-umbral extensions of Stirling numbers and Dobinski-like formulas,* Adv. Stud. Contemp. Math. (Kyungshang) **12** (2006), no. 1, 73-100.
13. H. Y. Lee, *On polynomials related to the Stirling numbers and the Bell polynomials associated with p-adic integral on $\mathbb{Z}_p$,* Proc. Jangjeon Math. Soc. **19** (2016), no. 4, 673-683.

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